

# Asymptotically polynomial solutions of difference equations of neutral type

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## Abstract

Asymptotic properties of solutions of difference equation of the form

$$\Delta^m(x_n + u_n x_{n+k}) = a_n f(n, x_{\sigma(n)}) + b_n$$

are studied. We give sufficient conditions under which all solutions, or all solutions with polynomial growth, or all nonoscillatory solutions are asymptotically polynomial. We use a new technique which allows us to control the degree of approximation.

**Key words:** difference equation, neutral equation, asymptotic behavior, asymptotically polynomial solution, nonoscillatory solution.

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## 1 Introduction

Let  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{R}$  denote the set of positive integers, all integers and real numbers respectively. Let  $m \in \mathbb{N}$ ,  $k \in \mathbb{Z}$ . We consider asymptotic properties of solutions of difference equations of the form

$$\Delta^m(x_n + u_n x_{n+k}) = a_n f(n, x_{\sigma(n)}) + b_n \quad (\text{E})$$

$$u_n, a_n, b_n \in \mathbb{R}, \quad f : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}, \quad \sigma : \mathbb{N} \rightarrow \mathbb{Z}, \quad \sigma(n) \rightarrow \infty, \quad u_n \rightarrow c \in \mathbb{R}, \quad |c| \neq 1.$$

By a solution of (E) we mean a sequence  $x : \mathbb{N} \rightarrow \mathbb{R}$  satisfying (E) for all large  $n$ . Asymptotic properties of solutions of neutral difference equations were investigated by many authors. These studies tend in several directions. For example, the papers [3], [15], [17] and [25] are devoted to the classification of solutions. In [8], [9], [11] and [24] were studied solutions with prescribed asymptotic behavior. In [1], [2], [10], [20] were investigated oscillatory solutions. Asymptotically polynomial solutions were studied in [16], [18], [21], [22]. Asymptotically polynomial solutions were also studied

in continuous case, see for example [5], [7], [19].

Thandapani, Arul and Raja in [21] establish conditions under which for any nonoscillatory solution  $x$  of the equation

$$\Delta^2(x_n + px_{n+k}) = f(n, x_{n+l}) \quad (1)$$

there exists a constant  $a$  such that

$$x_n = an + o(n).$$

In [16], there are given conditions under which any nonoscillatory solution  $x$  of (1) has an asymptotic behavior

$$x_n = an + b + o(1).$$

M. Migda, in [18], establish conditions under which for any nonoscillatory solution  $x$  of (E) there exists a constant  $a$  such that

$$x_n = an^{m-1} + o(n^{m-1}).$$

In this paper, in Theorem 1, we extend these results in the following way. Let  $s \in (-\infty, m-1]$  and let  $p$  be a nonnegative integer such that  $s \leq p \leq m-1$ . We establish conditions under which any solution, or any solution with polynomial growth, or any nonoscillatory solution  $x$  has an asymptotic behavior

$$x_n = a_{m-1}n^{m-1} + a_{m-2}n^{m-2} + \cdots + a_p n^p + o(n^s)$$

for some fixed real  $a_{m-1}, a_{m-2}, \dots, a_p$ .

The idea of the proof is as follows. Let  $z$  be a sequence defined by

$$z_n = x_n + u_n x_{n+k}. \quad (2)$$

Using  $z$  we can write equation (E) in the form

$$\Delta^m z_n = a_n f(n, x_{\sigma(n)}) + b_n. \quad (3)$$

Let  $s$  be a real number such that  $s \leq m-1$ . Assume that

$$\sum_{n=1}^{\infty} n^{m-1-s} |a_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} n^{m-1-s} |b_n| < \infty.$$

Using a Bihari type lemma and some additional assumptions, we show that (3) implies

$$\sum_{n=1}^{\infty} n^{m-1-s} |\Delta^m z_n| < \infty. \quad (4)$$

Next we use the result from [12], which states that if  $\Delta^m z$  is asymptotically zero, then  $z$  is asymptotically polynomial. More precisely, we show that (4) implies

$$z_n = \varphi(n) + o(n^s) \quad (5)$$

where  $\varphi$  is a polynomial sequence such that  $\deg \varphi < m$ . Finally, using our Lemma 3.5, we show that

$$x_n = \psi(n) + o(n^s) \quad (6)$$

for certain polynomial sequence  $\psi$  such that  $\deg \psi < m$ . In the last section we show, that if  $s = q$  is a nonnegative integer, then (6) may be replaced by a stronger condition

$$x_n = \psi(n) + w_n, \quad \Delta^k w_n = o(n^{q-k}) \quad \text{for } k = 0, 1, \dots, q.$$

The paper is organized as follows. In Section 2, we introduce notation and terminology. Section 3 is devoted to the proof of Lemma 3.5. In Section 4, we obtain Theorem 1, which is the main result of this paper. The proof of Theorem 1 is based on three lemmas: Lemma 3.5, Lemma 4.1, and Lemma 4.2. In Section 5, we obtain a result analogous to Theorem 1, but we replace the spaces of asymptotically polynomial sequences by the spaces of regularly asymptotically polynomial sequences (see (7)).

## 2 Notation and terminology

By  $\text{SQ}$  we denote the space of all sequences  $x : \mathbb{N} \rightarrow \mathbb{R}$ . If  $p, k \in \mathbb{Z}$ ,  $k \geq p$  then

$$\mathbb{N}(p, k) = \{p, p+1, \dots, k\}, \quad \mathbb{N}(p) = \{p, p+1, \dots\}.$$

For  $m \in \mathbb{N}(0)$ , we define

$$\text{Pol}(m-1) = \text{Ker } \Delta^m = \{x \in \text{SQ} : \Delta^m x = 0\}.$$

Then  $\text{Pol}(m-1)$  is the space of all polynomial sequences of degree less than  $m$ . Note that

$$\text{Pol}(-1) = \text{Ker } \Delta^0 = 0$$

is the zero space. For  $x, y \in \text{SQ}$ , we define the product  $xy$  by  $(xy)(n) = x_n y_n$  for any  $n$ . Moreover,  $|x|$  denotes the sequence defined by  $|x|(n) = |x_n|$  for any  $n$ .

We use the symbols "big O" and "small o" in the usual sense but for  $a \in \text{SQ}$  we also regard  $o(a)$  and  $O(a)$  as subspaces of  $\text{SQ}$ . More precisely

$$o(1) = \{x \in \text{SQ} : x_n \rightarrow 0\}, \quad O(1) = \{x \in \text{SQ} : x \text{ is bounded}\}$$

$$o(a) = ao(1) = \{ax : x \in o(1)\}, \quad O(a) = aO(1) = \{ax : x \in O(1)\}.$$

For a subset  $X$  of  $\text{SQ}$ , let

$$\Delta^m X = \{\Delta^m x : x \in X\}, \quad \Delta^{-m} X = \{z \in \text{SQ} : \Delta^m z \in X\}$$

denote respectively the image and the inverse image of  $X$  under the map  $\Delta^m : \text{SQ} \rightarrow \text{SQ}$ . Now, we can define spaces of asymptotically polynomial sequences and regularly asymptotically polynomial sequences

$$\text{Pol}(m-1) + o(n^s), \quad \text{Pol}(m-1) + \Delta^{-k} o(1), \quad (7)$$

where  $s \in (-\infty, m-1]$  and  $k \in \mathbb{N}(0, m-1)$ . Moreover, let

$$o(n^{-\infty}) = \bigcap_{s \in \mathbb{R}} o(n^s) = \bigcap_{k=1}^{\infty} o(n^{-k}), \quad O(n^{\infty}) = \bigcup_{s \in \mathbb{R}} O(n^s) = \bigcup_{k=1}^{\infty} O(n^k).$$

Note that the condition  $\limsup \sqrt[n]{|a_n|} < 1$  or  $\limsup \frac{|a_{n+1}|}{|a_n|} < 1$  implies  $a \in o(n^{-\infty})$ .

Let  $x, u \in \text{SQ}$  and  $k \in \mathbb{Z}$ . We say that  $x$  is nonoscillatory if  $x_n x_{n+1} \geq 0$  for large  $n$ . If  $x_n x_{n+k} \geq 0$  for large  $n$  we say that  $x$  is  $k$ -nonoscillatory. If  $x_n u_n x_{n+k} \geq 0$  for large  $n$  we say that  $x$  is  $(u, k)$ -nonoscillatory.

**Remark 2.1.** *If  $\liminf u_n > 0$ , then a sequence  $x$  is  $(u, k)$ -nonoscillatory if and only if it is  $k$ -nonoscillatory. If  $\limsup u_n < 0$ , then  $x \in \text{SQ}$  is  $(u, k)$ -nonoscillatory if and only if  $-x$  is  $k$ -nonoscillatory. Every nonoscillatory sequence  $x$  is also  $k$ -nonoscillatory for any  $k \in \mathbb{Z}$ .*

Let  $X$  be a metric space. A function  $g : X \rightarrow \mathbb{R}$  is called locally bounded if for any  $x \in X$  there exists a neighborhood  $U$  of  $x$  such that the restriction  $g|_U$  is bounded.

**Remark 2.2.** *If  $X$  is a closed subset of  $\mathbb{R}$ , then a function  $g : X \rightarrow \mathbb{R}$  is locally bounded if and only if it is bounded on every bounded subset of  $X$ . On the other hand if, for example,  $h : (0, \infty) \rightarrow \mathbb{R}$  is given by  $g(t) = t^{-1}$ , then  $g$  is locally bounded and  $g|(0, 1)$  is unbounded.*

Let  $f : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : [0, \infty) \rightarrow [0, \infty)$ , and  $p \in \mathbb{R}$ . We say that  $f$  is  $(g, p)$ -bounded if

$$|f(n, t)| \leq g\left(\frac{|t|}{n^p}\right)$$

for any  $(n, t) \in \mathbb{N} \times \mathbb{R}$ .

We say that a sequence  $x$  is of polynomial growth if  $x \in O(n^{\infty})$ .

### 3 Associated sequences

In this section we assume that  $x, u, z \in \text{SQ}$ ,  $k \in \mathbb{Z}$ ,  $\lim u_n = c \in \mathbb{R}$ ,  $|c| \neq 1$  and

$$z_n = x_n + u_n x_{n+k}, \quad \text{for } n \geq \max(0, -k).$$

This section is devoted to the proof of Lemma 3.5. In this lemma, we establish conditions under which, for a given real  $\alpha$ , the condition  $z \in \text{Pol}(m) + o(n^{\alpha})$  implies

$$x \in \text{Pol}(m) + o(n^{\alpha}).$$

Lemma 3.5 extends [16, Lemma 4] and will be used in the proof of Theorem 1.

**Lemma 3.1.** Assume  $x$  is bounded and  $z$  is convergent. Then  $x$  is convergent and

$$(1 + c) \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n.$$

*Proof.* See Lemma 1 in [16]. □

**Remark 3.1.** Boundedness of  $x$  cannot be omitted in Lemma 3.1. For example, if  $x_n = 2^n$ ,  $u_n = -2^{-1}$  and  $k = 1$ , then  $z_n = 0$  for any  $n$  and  $x$  is divergent. However, see the next lemma.

**Lemma 3.2.** Assume one of the following conditions is satisfied

$$(a) \quad |c| < 1 \quad \text{and} \quad k \leq 0, \quad (b) \quad |c| > 1 \quad \text{and} \quad k \geq 0.$$

Then boundedness of the sequence  $z$  implies boundedness of  $x$ .

*Proof.* Assume (a) and the sequence  $z$  is bounded. Choose  $b > 0$  such that  $|z_n| \leq b$  for all  $n$ . Choose a number  $\beta$  such that  $|c| < \beta < 1$ . Let  $r = -k$ . Then  $r \geq 0$  and there exists  $n_0 \geq r$  such that  $|u_n| < \beta$  for  $n \geq n_0$ . Let

$$K = \max(|x_0|, \dots, |x_{n_0}|), \quad n \in \mathbb{N}(n_0).$$

There exists  $m \in \mathbb{N}(0)$  such that

$$0 \leq n - mr \leq n_0, \quad n - (m - 1)r > n_0.$$

Since  $x_n = z_n - u_n x_{n-r}$ , we obtain

$$|x_n| \leq b + |u_n| |x_{n-r}| < b + \beta |x_{n-r}|.$$

Similarly  $|x_{n-r}| < b + \beta |x_{n-2r}|$ . Hence

$$|x_n| < b + \beta b + \beta^2 |x_{n-2r}|$$

and so on. After  $m$  steps we obtain

$$|x_n| < b(1 + \beta + \beta^2 + \dots + \beta^{m-1}) + \beta^m |x_{n-mr}|.$$

Since  $\beta \in (0, 1)$  and  $n - mr \leq n_0$ , we have  $\beta^m |x_{n-mr}| < K$ . Hence

$$|x_n| < \frac{b}{1 - \beta} + K.$$

So, the sequence  $(x_n)$  is bounded.

Now, assume (b). Let

$$v_n = \frac{1}{u_n}, \quad c' = \frac{1}{c}, \quad y_n = u_n x_{n+k}.$$

Then  $|c'| < 1$ ,  $\lim v_n = c'$  and  $y_n + v_n y_{n-k} = u_n x_{n+k} + x_n = z_n$ . Hence, by first part of the proof, the sequence  $y$  is bounded. Therefore the sequence  $x = z - y$  is bounded too. The proof is complete. □

Lemma 3.2 extends [16, Lemma 2].

**Lemma 3.3.** *If  $x \in O(n^\infty)$  and  $z$  is bounded, then  $x$  is bounded.*

*Proof.* By Lemma 3.2, we can assume that one of the following conditions is satisfied

$$(a) \quad |c| < 1 \quad \text{and} \quad k > 0, \quad (b) \quad |c| > 1 \quad \text{and} \quad k < 0.$$

Assume (a) and choose  $M > 1$  such that  $|z_n| \leq M$  for all  $n$ . Choose a number  $\rho$  such that  $|c| < \rho < 1$ . There exists an index  $n_1$  such that  $|u_n| < \rho$  for  $n \geq n_1$ . Then

$$|z_n - x_n| = |u_n||x_{n+k}| < \rho|x_{n+k}| \quad (8)$$

for  $n \geq n_1$ . Let  $r = \rho^{-1}$ . Then  $r > 1$  and, by (8),

$$|x_{n+k}| > r|z_n - x_n|$$

for  $n \geq n_1$ . Choose a constant  $N$  such that

$$N > \frac{1}{r-1}. \quad (9)$$

Assume the sequence  $(x_n)$  is unbounded. Then there exists  $p \geq n_1$  such that

$$|x_p| \geq (N+1)M. \quad (10)$$

Since  $|z_p| \leq M$ , by (10), we have  $|z_p - x_p| \geq NM$ . Then

$$|x_{p+k}| > r|z_p - x_p| \geq rNM.$$

The condition  $|z_{p+k}| \leq M$  implies

$$|z_{p+k} - x_{p+k}| \geq rNM - M = (rN - 1)M.$$

Hence

$$|x_{p+2k}| > r|z_{p+k} - x_{p+k}| \geq r(rN - 1)M. \quad (11)$$

Since  $|z_{p+2k}| \leq M$ , we obtain

$$|z_{p+2k} - x_{p+2k}| \geq r(rN - 1)M - M = (r(rN - 1) - 1)M.$$

If

$$a_1 = rN, \quad a_2 = r(a_1 - 1), \quad \dots, \quad a_{n+1} = r(a_n - 1),$$

then, as in (11), we have

$$|x_{p+nk}| \geq a_n M \quad (12)$$

for  $n \geq 1$ . Moreover,

$$a_2 = r(a_1 - 1) = r^2N - r, \quad a_3 = r(a_2 - 1) = r^3N - r^2 - r$$

and so on. Hence, for  $n \geq 1$ , we obtain

$$\begin{aligned} a_n &= r^n N - (r^{n-1} + r^{n-2} + \cdots + r + 1) + 1 \\ &= r^n N - \frac{r^n - 1}{r - 1} + 1 = \left( N - \frac{1}{r - 1} \right) r^n + \frac{1}{r - 1} + 1. \end{aligned}$$

Let

$$a = N - \frac{1}{r - 1}, \quad b = \frac{1}{r - 1} + 1.$$

By (9),  $a > 0$ . Since  $r > 1$ , we have  $b > 0$ . Moreover, by (12),

$$|x_{p+nk}| \geq ar^n + b$$

for  $n \geq 1$ . Since  $x \in O(n^\infty)$ , there exists a number  $\alpha > 1$  such that  $x_n = O(n^\alpha)$ . There exist  $w \in (0, \infty)$  and  $m_0 \in \mathbb{N}(0)$  such that

$$(p + nk)^\alpha < wn^\alpha$$

for  $n \geq m_0$ . Then

$$\frac{x_{p+nk}}{(p + nk)^\alpha} > \frac{ar^n + b}{(p + nk)^\alpha} > \frac{a}{w} \frac{r^n}{n^\alpha}$$

for  $n \geq m_0$ . It is impossible since  $r > 1$  and  $x_n = O(n^\alpha)$ . Hence, the sequence  $(x_n)$  is bounded. Now assume (b) and  $x_n = O(n^\alpha)$ . Let

$$v_n = \frac{1}{u_n}, \quad c' = \frac{1}{c}, \quad y_n = u_n x_{n+k}.$$

Then

$$|c'| < 1, \quad \lim v_n = c', \quad y_n = O(n^\alpha), \quad y_n + v_n y_{n-k} = u_n x_{n+k} + x_n = z_n$$

and by the first part of the proof the sequence  $(y_n)$  is bounded. Hence, the sequence  $x_n = z_n - y_n$  is bounded too.  $\square$

**Lemma 3.4.** *Let  $\alpha \in \mathbb{R}$ . Assume  $k(|c| - 1) \geq 0$  or  $x \in O(n^\infty)$ . Then*

- (1) *if  $z_n = O(n^\alpha)$ , then  $x_n = O(n^\alpha)$ ,*
- (2) *if  $z_n = o(n^\alpha)$ , then  $x_n = o(n^\alpha)$ .*

*Proof.* Assume  $\alpha = 0$ . If  $k(|c| - 1) \geq 0$ , then the result follows from Lemma 3.2 and Lemma 3.1. If  $x \in O(n^\infty)$ , then by Lemma 3.3 boundedness of  $z$  implies boundedness of  $x$ . Moreover, by Lemma 3.3 and Lemma 3.1, convergence of  $z$  implies convergence of  $x$ . Now assume  $\alpha$  is an arbitrary real number. By the equality

$$\frac{z_n}{n^\alpha} = \frac{x_n}{n^\alpha} + u_n \frac{(n+k)^\alpha}{n^\alpha} \frac{x_{n+k}}{(n+k)^\alpha} = \frac{x_n}{n^\alpha} + u_n \left(1 + \frac{k}{n}\right)^\alpha \frac{x_{n+k}}{(n+k)^\alpha}$$

and the equality

$$\lim_{n \rightarrow \infty} \left(1 + \frac{k}{n}\right)^\alpha = 1$$

we see that the result is a consequence of the first part of the proof.  $\square$

Now, we are ready to state and prove the main result of this section.

**Lemma 3.5.** *Assume  $k(|c| - 1) \geq 0$  or  $x \in O(n^\infty)$ . Let  $m \in \mathbb{N}(0)$ ,  $\alpha \in \mathbb{R}$ , and*

$$u_n = c + o(n^{\alpha-m}).$$

*Then the condition  $z \in \text{Pol}(m) + o(n^\alpha)$  implies  $x \in \text{Pol}(m) + o(n^\alpha)$ .*

*Proof.* If  $\alpha > m$ , then

$$\text{Pol}(m) + o(n^\alpha) = o(n^\alpha)$$

and the assertion follows from Lemma 3.4. Assume  $\alpha \leq m$ . For  $n \geq \max(0, -k)$ , let

$$z'_n = x_n + cx_{n+k}.$$

We will show, by induction on  $m$ , that

$$z' \in \text{Pol}(m) + o(n^\alpha) \implies x \in \text{Pol}(m) + o(n^\alpha). \quad (13)$$

For  $m = -1$  this assertion follows from Lemma 3.4. Assume it is true for certain  $m \geq -1$  and let

$$z' \in \text{Pol}(m+1) + o(n^\alpha).$$

Then there exist  $a \in \mathbb{R}$  and  $w \in \text{Pol}(m) + o(n^\alpha)$  such that

$$z'_n = an^{m+1} + w_n.$$

Since

$$a = \frac{a}{1+c} + \frac{ca}{1+c}, \quad (n+k)^{m+1} = n^{m+1} + r_n, \quad r \in \text{Pol}(m)$$

we obtain

$$\begin{aligned} w_n &= z'_n - an^{m+1} = x_n - \frac{a}{1+c}n^{m+1} + cx_{n+k} - \frac{ca}{1+c}n^{m+1} \\ &= \left( x_n - \frac{a}{1+c}n^{m+1} \right) + c \left( x_{n+k} - \frac{a}{1+c}(n+k)^{m+1} + \frac{a}{1+c}r_n \right). \end{aligned}$$

Let

$$v_n = x_n - \frac{a}{1+c}n^{m+1}.$$

Then

$$w_n - \frac{ca}{1+c}r_n = v_n + cv_{n+k}.$$

Since  $r, w \in \text{Pol}(m) + o(n^\alpha)$  we obtain

$$\left( w - \frac{ca}{1+c}r \right) \in \text{Pol}(m) + o(n^\alpha).$$

The condition  $z' \in \text{Pol}(m) + o(n^\alpha)$  implies  $z' = O(n^m)$ . Hence, by Lemma 3.4,  $x_n = O(n^m)$ . Therefore  $v \in O(n^\infty)$  and, by inductive hypothesis,

$$v \in \text{Pol}(m) + o(n^\alpha).$$



By the equality

$$x_n = v_n + \frac{a}{1+c} n^{m+1},$$

we have

$$x \in \text{Pol}(m+1) + o(n^\alpha).$$

Now, assume

$$z \in \text{Pol}(m) + o(n^\alpha).$$

Since  $\alpha \leq m$  we have  $z_n = O(n^m)$  and, by Lemma 3.4,  $x_n = O(n^m)$ . Hence  $x_{n+k} = O(n^m)$  and from the condition  $u_n = c + o(n^{\alpha-m})$  we obtain

$$z'_n - z_n = (c - u_n)x_{n+k} = n^\alpha \frac{c - u_n}{n^{\alpha-m}} \frac{x_{n+k}}{n^m} = n^\alpha o(1)O(1) = o(n^\alpha).$$

Hence the condition  $z \in \text{Pol}(m) + o(n^\alpha)$  implies

$$z'_n = z_n + (z'_n - z_n) \in \text{Pol}(m) + o(n^\alpha) + o(n^\alpha) = \text{Pol}(m) + o(n^\alpha)$$

and the result follows from the first part of the proof.  $\square$

## 4 Asymptotically polynomial solutions 1

In this section, in Theorem 1, we obtain our main result. First, in Lemma 4.1, we obtain a certain discrete version of the Bihari's lemma. This version is similar to Theorem 1 in [4] but we do not assume the continuity of  $g$ .

**Lemma 4.1.** *Assume  $a, w$  are nonnegative sequences,  $p \in \mathbb{N}$ ,*

$$g : [0, \infty) \rightarrow [0, \infty), \quad 0 \leq \lambda < M, \quad g(\lambda) > 0,$$

$$\sum_{k=0}^{\infty} a_k \leq \int_{\lambda}^M \frac{dt}{g(t)}, \tag{14}$$

$$w_n \leq \lambda + \sum_{k=p}^{n-1} a_k g(w_k)$$

for  $n \geq p$  and  $g$  is nondecreasing. Then  $w_n \leq M$  for  $n \geq p$ .

*Proof.* For  $n \geq p$ , let

$$s_n = \lambda + \sum_{k=p}^{n-1} a_k g(w_k).$$

Then, for  $n \geq p$ , we have  $\Delta s_n = s_{n+1} - s_n = a_n g(w_n) \leq a_n g(s_n)$  and

$$\int_{s_n}^{s_{n+1}} \frac{dt}{g(t)} \leq \int_{s_n}^{s_{n+1}} \frac{dt}{g(s_n)} = \frac{\Delta s_n}{g(s_n)} \leq a_n.$$

Therefore, using (14), we have

$$\int_{\lambda}^{s_n} \frac{dt}{g(t)} = \sum_{k=p}^{n-1} \int_{s_k}^{s_{k+1}} \frac{dt}{g(t)} \leq \sum_{k=p}^{n-1} a_k \leq \int_{\lambda}^M \frac{dt}{g(t)}.$$

Since  $g$  is positive on  $[\lambda, \infty)$ , we obtain  $s_n \leq M$ . Hence

$$w_n \leq s_n \leq M$$

for  $n \geq p$ . The proof is complete.  $\square$

In the proof of Theorem 1 we also use the following two lemmas.

**Lemma 4.2.** Assume  $m \in \mathbb{N}(1)$ ,  $z \in \text{SQ}$ ,  $s \in (-\infty, m-1]$  and

$$\sum_{n=1}^{\infty} n^{m-1-s} |\Delta^m z_n| < \infty.$$

Then  $z \in \text{Pol}(m-1) + o(n^s)$ .

*Proof.* The assertion follows from the proof of Theorem 2.1 in [12].  $\square$

**Lemma 4.3.** If  $x \in \text{SQ}$  and  $m, n_0 \in \mathbb{N}$ , then there exists  $L > 0$  such that

$$|x_n| \leq n^{m-1} \left( L + \sum_{i=n_0}^{n-1} |\Delta^m x_i| \right) \quad \text{for } n \geq n_0.$$

*Proof.* See [13, Lemma 7.3].  $\square$

Now we are ready to prove our main result.

**Theorem 1.** Assume  $m \in \mathbb{N}$ ,  $k \in \mathbb{Z}$ ,  $c, s, p \in \mathbb{R}$ ,  $|c| \neq 1$ ,  $s \leq m-1$ ,  $a, b, u \in \text{SQ}$ ,

$$f : \mathbb{N} \times \mathbb{R} \rightarrow \mathbb{R}, \quad g : [0, \infty) \rightarrow [0, \infty), \quad \sigma : \mathbb{N} \rightarrow \mathbb{Z}, \quad \sigma(n) \rightarrow \infty,$$

$$\sum_{n=1}^{\infty} n^{m-1-s} |a_n| < \infty, \quad \sum_{n=1}^{\infty} n^{m-1-s} |b_n| < \infty, \quad u_n = c + o(n^{s+1-m}),$$

$x$  is a solution of (E) and one of the following conditions is satisfied:

(a)  $g$  is nondecreasing,  $f$  is  $(g, m-1)$ -bounded,  $\sigma(n) \leq n$  for large  $n$ ,

$$\int_1^{\infty} \frac{dt}{g(t)} = \infty,$$

and  $x$  is  $(u, k)$ -nonoscillatory,

(b)  $g$  is locally bounded,  $f$  is  $(g, p)$ -bounded,  $x \circ \sigma = O(n^p)$  and the following alternative is satisfied:  $k(|c| - 1) \geq 0$  or  $x \in O(n^\infty)$  or  $x$  is  $(u, k)$ -nonoscillatory,

(c)  $f$  is bounded and the following alternative is satisfied:  $k(|c| - 1) \geq 0$  or  $x \in O(n^\infty)$  or  $x$  is  $(u, k)$ -nonoscillatory.

Then

$$x \in \text{Pol}(m-1) + o(n^s). \quad (15)$$

*Proof.* Let  $z \in \text{SQ}$ ,

$$z_n = x_n + u_n x_{n+k}$$

for large  $n$ . Assume (b). Since  $x \circ \sigma = O(n^p)$  and  $f$  is  $(g, p)$ -bounded, we see that the sequence  $(f(n, x_{\sigma(n)}))$  is bounded. Hence

$$\sum_{n=1}^{\infty} n^{m-1-s} |\Delta^m z_n| < \infty$$

and, by Lemma 4.2, we have  $z \in \text{Pol}(m-1) + o(n^s)$ . If  $x$  is  $(u, k)$ -nonoscillatory, then

$$|z_n| = |x_n + u_n x_{n+k}| = |x_n| + |u_n x_{n+k}|$$

for large  $n$ . Hence

$$|x_n| \leq |z_n| \quad (16)$$

for large  $n$ . Therefore  $x \in O(n^\infty)$ . Now, using Lemma 3.5, we obtain (15). The proof in the case (c) is analogous.

Assume (a). There exists an index  $n_0$  such that

$$|x_n| \leq |z_n|, \quad \sigma(n) \geq 1, \quad \sigma(n) \leq n$$

and (E) is satisfied for  $n \geq n_0$ . Choose an index  $n_1 \geq n_0$  such that  $\sigma(n) \geq n_0$  for  $n \geq n_1$ . By Lemma 4.3, there exists a positive constant  $L$  such that

$$\frac{|z_n|}{n^{m-1}} \leq L + \sum_{j=1}^{n-1} |\Delta^m z_j| \quad (17)$$

for any  $n$ . Let

$$L_1 = L + \sum_{j=1}^{n_1} |\Delta^m z_j|, \quad L_2 = L_1 + \sum_{j=1}^{\infty} |b_j|.$$

If  $n \geq n_1$ , then, using (17), (E),  $(g, m-1)$ -boundedness of  $f$ , and (16), we obtain

$$\begin{aligned} \frac{|z_{\sigma(n)}|}{n^{m-1}} &\leq \frac{|z_{\sigma(n)}|}{\sigma(n)^{m-1}} \leq L + \sum_{j=1}^{\sigma(n)-1} |\Delta^m z_j| \leq L + \sum_{j=1}^{n-1} |\Delta^m z_j| \\ &\leq L_1 + \sum_{j=n_1}^{n-1} |\Delta^m z_j| \leq L_2 + \sum_{j=n_1}^{n-1} |a_j| g\left(\frac{|x_{\sigma(j)}|}{j^{m-1}}\right) \leq L_2 + \sum_{j=n_1}^{n-1} |a_j| g\left(\frac{|z_{\sigma(j)}|}{j^{m-1}}\right). \end{aligned}$$

By Lemma 4.1, the sequence  $(z_{\sigma(n)}/n^{m-1})$  is bounded. Hence, by (16),

$$x \circ \sigma = O(n^{m-1}).$$

Therefore, taking  $p = m-1$  in (b), we obtain (15). The proof is complete.  $\square$

**Remark 4.1.** The condition  $x \in O(n^\infty)$  is not a consequence of  $x \circ \sigma \in O(n^\infty)$ . For example, if  $x_n = e^n$ ,  $\sigma(n) = \lfloor \log n \rfloor$  (integer part of  $\log n$ ), then  $x \circ \sigma = O(n)$  and  $x \notin O(n^\infty)$ .

**Remark 4.2.** If the sequence  $u$  is nonnegative, then the class of  $(u, k)$ -nonoscillatory sequences is larger than the class of nonoscillatory sequences. Moreover, if

$$n_1 = \min\{n \in \mathbb{N} : \sigma(i) \geq 1 \text{ for } i \geq n\}$$

and we define a full solution of (E) as a sequence  $x$  such that (E) is satisfied for all  $n \geq \max(n_1, -k)$ , then the set of full solutions is a subset of the set of all solutions. Hence Theorem 1 covers the case of full solutions and, assuming  $u$  is nonnegative, the case of nonoscillatory solutions.

**Lemma 4.4.** If  $m \in \mathbb{N}(0)$ , then

$$\text{Pol}(m-1) + o(n^{-\infty}) = \bigcap_{k=1}^{\infty} (\text{Pol}(m-1) + o(n^{-k})).$$

*Proof.* Let  $P = \text{Pol}(m-1)$  and

$$x \in \bigcap_{k=1}^{\infty} (P + o(n^{-k})).$$

Then  $x \in P + o(1)$  and  $x = \varphi + u$  for some  $\varphi \in P$  and  $u \in o(1)$ . Since  $P \cap o(1) = 0$ , the sequences  $\varphi$  and  $u$  are unique. Let  $k \in \mathbb{N}$ . Then  $x \in P + o(n^{-k})$  and by uniqueness of  $u \in o(1)$  we have  $u \in o(n^{-k})$ . Hence  $u \in o(n^{-\infty})$  and we obtain

$$\bigcap_{k=1}^{\infty} (P + o(n^{-k})) \subset P + o(n^{-\infty}).$$

The inverse inclusion is obvious. □

**Corollary 4.1.** Assume all conditions of Theorem 1 are satisfied and  $a, b \in o(n^{-\infty})$ . Then

$$x \in \text{Pol}(m-1) + o(n^{-\infty}).$$

*Proof.* The assertion is a consequence of Theorem 1 and Lemma 4.4. □

## 5 Asymptotically polynomial solutions 2

In this section, in Theorem 2, we obtain a result analogous to Theorem 1. We replace the spaces of asymptotically polynomial sequences by the spaces of regularly asymptotically polynomial sequences. The study of regularly asymptotically polynomial sequences

$$\text{Pol}(m) + \Delta^{-q}o(1), \quad q \in \mathbb{N}(0, m)$$

is motivated by a special case  $\text{Pol}(m) + \Delta^{-m}o(1)$ . By Remark 5.2, the condition

$$z \in \text{Pol}(m) + \Delta^{-m}o(1)$$

is equivalent to the convergence of the sequence  $\Delta^m z$  and the condition

$$\lim_{n \rightarrow \infty} \Delta^m z_n = \lambda$$

is equivalent to the condition

$$\lim_{n \rightarrow \infty} \frac{p! \Delta^{m-p} z_n}{n^p} = \lambda \quad \text{for any } p \in \mathbb{N}(0, m). \quad (18)$$

Convergence of the sequence  $\Delta^m z_n$  is comparatively easy to verify and condition (18) appears in many papers, see for example [6], [14], [18], [23] or the proof of Theorem 3.1 in [22].

In the next lemma, we establish some basic properties of spaces of regularly asymptotically polynomial sequences.

**Lemma 5.1.** *Assume  $m \in \mathbb{N}$ ,  $k \in \mathbb{N}(0, m)$  and  $x \in \text{SQ}$ . Then*

- (a)  $x \in \Delta^{-m}o(1) \iff \Delta^p x \in o(n^{m-p})$  for every  $p \in \mathbb{N}(0, m)$ ,
- (b)  $x \in \text{Pol}(m) + \Delta^{-k}o(1) \iff \Delta^p x \in \text{Pol}(m-p) + o(n^{k-p})$  for any  $p \in \mathbb{N}(0, k)$ .
- (c)  $\text{Pol}(m-1) \subset \Delta^{-m}o(1) \subset o(n^m)$ ,  $o(n^m) \setminus \Delta^{-m}o(1) \neq \emptyset$ .
- (d)  $\Delta^{-m}o(1) = \{z \in o(n^m) : \Delta^p z \in o(n^{m-p}) \text{ for any } p \in \mathbb{N}(0, m)\}$ .

*Proof.* (a) If  $x \in \Delta^{-m}o(1)$ , then  $\Delta^m x = o(1)$  and

$$\frac{\Delta \Delta^{m-1} x_n}{\Delta n} = \Delta^m x_n = o(1).$$

By the Stolz-Cesaro theorem  $\Delta^{m-1} x_n = o(n)$ . Hence

$$\frac{\Delta \Delta^{m-2} x_n}{\Delta n^2} = \frac{n \Delta \Delta^{m-2} x_n}{n \Delta n^2} = \frac{\Delta^{m-1} x_n}{n} \frac{n}{\Delta n^2} \rightarrow 0.$$

Again, by the Stolz-Cesaro theorem,  $\Delta^{m-2} x_n = o(n^2)$ . Analogously  $\Delta^{m-3} x_n = o(n^3)$  and so on. Inverse implication is obvious.

(b) and (d) are consequences of (a).

(c) The inclusion

$$\text{Pol}(m-1) \subset \Delta^{-m}o(1)$$

is obvious. The inclusion

$$\Delta^{-m}o(1) \subset o(n^m)$$

is a consequence of (a). If  $a_n = (-1)^n$ , then

$$\Delta^m a_n = 2^m (-1)^{m+n} \notin o(1).$$

Hence  $a \in o(n^m) \setminus \Delta^{-m}o(1)$ . □

**Remark 5.1.** Assume  $m \in \mathbb{N}(0)$ ,  $k \in \mathbb{N}(0, m)$ . If  $\text{Pol}(m, k)$  denotes the subspace of  $\text{Pol}(m)$  generated by sequences  $(n^m), (n^{m-1}), \dots, (n^k)$ , then

$$\text{Pol}(m) + o(n^k) = \text{Pol}(m, k) + o(n^k) \quad \text{and} \quad \text{Pol}(m, k) \cap o(n^k) = 0.$$

Hence,  $x \in \text{Pol}(m) + o(n^k)$  if and only if there exist constants  $c_m, \dots, c_k$  and a sequence  $w \in o(n^k)$  such that

$$x_n = c_m n^m + c_{m-1} n^{m-1} + \dots + c_k n^k + w_n.$$

Moreover, the constants  $c_m, \dots, c_k$  and the sequence  $w$  are unique and

$$x \in \text{Pol}(m) + \Delta^{-k} o(1) \iff \Delta^p w_n = o(n^{k-p}) \quad \text{for any } p \in \mathbb{N}(0, k).$$

If  $P(m, k)$  and  $D(m, k)$  denote the spaces defined by

$$P(m, k) = \text{Pol}(m) + o(n^k) \quad \text{and} \quad D(m, k) = \text{Pol}(m) + \Delta^{-k} o(1).$$

respectively, then we obtain a diagram

$$\begin{array}{ccccccccc} P(m, 0) & \longrightarrow & P(m, 1) & \longrightarrow & P(m, 2) & \longrightarrow & \dots & \longrightarrow & P(m, m) & \longrightarrow & P(m, m+1) \\ \uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\ D(m, 0) & \longrightarrow & D(m, 1) & \longrightarrow & D(m, 2) & \longrightarrow & \dots & \longrightarrow & D(m, m) & \longrightarrow & D(m, m+1) \end{array}$$

where arrows denote inclusions. Note that

$$D(m, 0) = \text{Pol}(m) + o(1) = P(m, 0)$$

and for  $k > m$  we have

$$P(m, k) = o(n^k), \quad D(m, k) = \Delta^{-k} o(1).$$

**Remark 5.2.** Assume  $m \in \mathbb{N}(0)$  and  $x \in \text{SQ}$ . If

$$x \in \text{Pol}(m) + \Delta^{-m} o(1), \tag{19}$$

then, by Lemma 5.1, the sequence  $\Delta^m x$  is convergent. On the other hand, if  $\lambda \in \mathbb{R}$  and

$$\Delta^m x = \lambda + o(1), \tag{20}$$

then taking  $w_n = \lambda n^m / m!$  we have  $\Delta^m(x - w) = \lambda + o(1) - \lambda = o(1)$ . Hence

$$x = w + (x - w) \in \text{Pol}(m) + \Delta^{-m} o(1).$$

Using the Stolz-Cesaro theorem one can show that condition (20) is equivalent to the condition

$$\lim_{n \rightarrow \infty} \frac{p! \Delta^{m-p} z_n}{n^p} = \lambda \quad \text{for any } p \in \mathbb{N}(0, m).$$

The next two lemmas are ‘regular’ versions of Lemmas 4.2 and 3.5.

**Lemma 5.2.** *Let  $m \in \mathbb{N}$ ,  $q \in \mathbb{N}(0, m-1)$ ,  $z \in \text{SQ}$  and*

$$\sum_{n=1}^{\infty} n^{m-q-1} |\Delta^m z_n| < \infty.$$

*Then  $z \in \text{Pol}(m-1) + \Delta^{-q}\mathbf{o}(1)$ .*

*Proof.* By Lemma 2.3 in [12], there exists  $w = \mathbf{o}(1)$  such that  $\Delta^m z = \Delta^{m-q}w$ . Choose  $x \in \text{SQ}$  such that  $\Delta^q x = w$ . Then  $x \in \Delta^{-q}\mathbf{o}(1)$  and

$$\Delta^m z = \Delta^{m-q}w = \Delta^{m-q}\Delta^q x = \Delta^m x.$$

Hence  $z - x \in \text{Pol}(m-1)$  and

$$z = z - x + x \in \text{Pol}(m-1) + \Delta^{-q}\mathbf{o}(1).$$

□

**Lemma 5.3.** *Let  $m \in \mathbb{N}(0)$ ,  $q \in \mathbb{N}(0, m)$  and  $u_n = c + \mathbf{o}(n^{-m})$ . Assume  $k(|c| - 1) \geq 0$  or  $x \in \mathbf{O}(n^\infty)$  and  $z \in \text{Pol}(m) + \Delta^{-q}\mathbf{o}(1)$ . Then*

$$x \in \text{Pol}(m) + \Delta^{-q}\mathbf{o}(1).$$

*Proof.* For  $n \geq n_0$  let  $z'_n = x_n + cx_{n+k}$ . We will show, by induction on  $q$ , that

$$z' \in \text{Pol}(m) + \Delta^{-q}\mathbf{o}(1) \implies x \in \text{Pol}(m) + \Delta^{-q}\mathbf{o}(1).$$

For  $q = 0$  this assertion follows from Lemma 3.5. Assume it is true for some  $q \geq 0$ . Let

$$z' \in \text{Pol}(m) + \Delta^{-(q+1)}\mathbf{o}(1), \quad z'' = \Delta z', \quad \text{and} \quad x'' = \Delta x.$$

Then

$$\begin{aligned} z''_n &= \Delta z'_n = \Delta(x_n + cx_{n+k}) = \Delta x_n + c\Delta x_{n+k} = x''_n + cx''_{n+k}, \\ z'' &= \Delta z' \in \Delta(\text{Pol}(m) + \Delta^{-(q+1)}\mathbf{o}(1)) = \text{Pol}(m-1) + \Delta^{-q}\mathbf{o}(1). \end{aligned}$$

If  $x \in \mathbf{O}(n^\infty)$ , then  $x'' = \Delta x \in \mathbf{O}(n^\infty)$ . By inductive hypothesis

$$x'' \in \text{Pol}(m-1) + \Delta^{-q}\mathbf{o}(1).$$

By equality  $x'' = \Delta x$  we obtain  $x \in \text{Pol}(m) + \Delta^{-(q+1)}\mathbf{o}(1)$ . Now, assume

$$z \in \text{Pol}(m) + \Delta^{-q}\mathbf{o}(1).$$

Then  $z_n = \mathbf{O}(n^m)$  and, by Lemma 3.4,  $x_n = \mathbf{O}(n^m)$ . Hence  $x_{n+k} = \mathbf{O}(n^m)$ . Since

$$u_n = c + \mathbf{o}(n^{-m}),$$

we have

$$z'_n - z_n = (c - u_n)x_{n+k} = \frac{c - u_n}{n^{-m}} \frac{x_{n+k}}{n^m} = \mathbf{o}(1).$$

Therefore

$$z' = z + (z' - z) \in \text{Pol}(m) + \Delta^{-q}\mathbf{o}(1) + \mathbf{o}(1) = \text{Pol}(m) + \Delta^{-q}\mathbf{o}(1).$$

Hence, by the first part of the proof, we obtain  $x \in \text{Pol}(m) + \Delta^{-q}\mathbf{o}(1)$ . □

**Theorem 2.** Assume all assumptions of Theorem 1 are satisfied and moreover let

$$s = q \in \mathbb{N}(0, m - 1] \quad \text{and} \quad u_n = c + o(n^{1-m}).$$

Then

$$x \in \text{Pol}(m - 1) + \Delta^{-q}o(1).$$

*Proof.* Repeat the proof of Theorem 1 replacing Lemma 4.2 and Lemma 3.5 by Lemma 5.2 and Lemma 5.3, respectively.  $\square$

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